
Notes on PFSS Extrapolation

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Abstract This is a documentation on the Stanford PFSS model. Detailed mathematical deductions are provided for the use of the model. Some brief documentation on PFSS-like models (SCS, HCCSSS, etc.) is also provided.

1. Basic Equation

The most common version of PFSS model (Hoeksema, 1984; Hoeksema and Scherrer, 1986; Wang and Sheeley, 1992) currently in use takes global radial Carrington synoptic maps as input. In these maps, photospheric fields are sampled on a heliographic coordinate, evenly spaced either in latitude or sine-latitude steps. If the field is purely potential, we have

$$\vec{B} = -\nabla\Psi, \tag{1}$$

where

$$\nabla^2\Psi = 0. \tag{2}$$

We assume the existence of a spherical "source surface" at a radius of R_s (usually at $2.5R_\odot$), beyond which all field lines are open and radial. The potential arises from both inside the inner boundary R_0 (photsphere, or $R_\odot$) and outside the outer boundary, or the source surface:

$$\Psi = \Psi_I + \Psi_O, \tag{3}$$

with

$$\Psi_I = \sum_{l=0}^{\infty} r^{-(l+1)} \sum_{m=-l}^{l} f_{I_{lm}} Y_{lm}(\theta,\phi), \tag{4}$$

$$\Psi_O = \sum_{l=0}^{\infty} r^{l} \sum_{m=-l}^{l} f_{O_{lm}} Y_{lm}(\theta,\phi). \tag{5}$$

Scale r in terms $R_\odot$ for Ψ_I and in terms of R_s for Ψ_O. Use the fact that

$$Y_{lm}(\theta,\phi) = k_{lm} P_l^m(\cos\theta) e^{im\phi}. \tag{6}$$

The real part of Ψ can be generalized from Equation (3) through (6):

$$\begin{aligned}\Psi &= R_0 \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\cos\theta)\left\{ g'_{lm}\cos m\phi \left[\left(\frac{R_0}{r}\right)^{l+1} + \frac{R_s}{R_0}\left(\frac{r}{R_s}\right)^{l} c_{lm}\right]\right. \\ &\qquad \left. + h'_{lm}\sin m\phi \left[\left(\frac{R_0}{r}\right)^{l+1} + \frac{R_s}{R_0}\left(\frac{r}{R_s}\right)^{l} d_{lm}\right]\right\}, \qquad (7)\end{aligned}$$

where g'_{lm}, h'_{lm}, c_{lm} and d_{lm} are the unknown coefficients. Note that the normalization of the spherical harmonics and associated Legendre fuctions can be tricky. We will simply present the normalization we adopted here and leave the detailed description to Section 2.

By definition, the field lines turn radial at the source surface. This means the field vector is purely radial at R_s, or rather, the potential is a constant on the source surface. Set this potential to 0, we then have

$$c_{lm} = d_{lm} = -\left(\frac{R_0}{R_s}\right)^{l+2} = c_l. \qquad (8)$$

Now our sole task is to determine g'_{lm} and h'_{lm}, using the inner boundary condition (photospheric field). Write B_r from Equation (1) specifically:

$$\begin{aligned}B_r(r,\theta,\phi) &= -\frac{\partial\Psi}{\partial r} = \sum_{lm} P_l^m(\cos\theta)(g'_{lm}\cos m\phi + h'_{lm}\sin m\phi) \\ &\qquad \left[(l+1)\left(\frac{R_0}{r}\right)^{l+2} - l\left(\frac{r}{R_s}\right)^{l-1} c_l\right]. \qquad (9)\end{aligned}$$

At inner boundary, we have

$$B_r(R_0,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\cos\theta)(g_{lm}\cos m\phi + h_{lm}\sin m\phi), \qquad (10)$$

where

$$g_{lm} = g'_{lm}\left[l+1+l\left(\frac{R_0}{R_s}\right)^{2l+1}\right], \qquad (11)$$

$$h_{lm} = h'_{lm}\left[l+1+l\left(\frac{R_0}{R_s}\right)^{2l+1}\right]. \qquad (12)$$

Now we make use of the orthogonal property of the associated Legendre function (in our convention)

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta\, P_l^m(\cos\theta) \overset{\cos}{\underset{\sin}{}} m\phi\, P_{l'}^{m'}(\cos\theta) \overset{\cos}{\underset{\sin}{}} m'\phi = \frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'}. \qquad (13)$$

Note when $m = 0$ Equation (13) holds for the $\cos m\phi$ case, while the $\sin m\phi$ case simply yields 0. An integration of Equation (10) then shows us how to obtain g and h. Here, h_{l0} is obviously 0.

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta \, B_r(R_0, \theta, \phi) P_l^m(\cos\theta) \begin{matrix}\cos \\ \sin\end{matrix} m\phi = \frac{4\pi}{2l+1} \begin{matrix} g_{lm} \\ h_{lm} \end{matrix}. \tag{14}$$

For a synoptic map $(X \times Y)$ in sine-latitude format, the Equation (14) becomes

$$\begin{pmatrix} g_{lm} \\ h_{lm} \end{pmatrix} = \frac{2l+1}{XY} \sum_{i=1}^{X} \sum_{j=1}^{Y} B_r(R_0, \theta_i, \phi_j) P_l^m(\cos\theta_i) \begin{matrix}\cos \\ \sin\end{matrix} m\phi_j. \tag{15}$$

Thus the potential is solved. There are other ways to compute g and h, as we will see in Section 3.

To conclude, we have the solutions in the following form.

$$\begin{aligned} B_r(r, \theta, \phi) = -\frac{\partial \Psi}{\partial r} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\cos\theta)(g_{lm} \cos m\phi + h_{lm} \sin m\phi) \times \\ \left(\frac{R_0}{r}\right)^{l+2} \left[l + 1 + l \left(\frac{r}{R_s}\right)^{2l+1} \right] \Big/ \left[l + 1 + l \left(\frac{R_0}{R_s}\right)^{2l+1} \right], \end{aligned} \tag{16}$$

$$\begin{aligned} B_\theta(r, \theta, \phi) = -\frac{1}{r} \frac{\partial \Psi}{\partial \theta} = -\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{\partial P_l^m(\cos\theta)}{\partial \theta}(g_{lm} \cos m\phi + h_{lm} \sin m\phi) \times \\ \left(\frac{R_0}{r}\right)^{l+2} \left[1 - \left(\frac{r}{R_s}\right)^{2l+1} \right] \Big/ \left[l + 1 + l \left(\frac{R_0}{R_s}\right)^{2l+1} \right], \end{aligned} \tag{17}$$

$$\begin{aligned} B_\phi(r, \theta, \phi) = -\frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial \phi} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\cos\theta)(g_{lm} \sin m\phi - h_{lm} \cos m\phi) \times \\ \left(\frac{R_0}{r}\right)^{l+2} \left[1 - \left(\frac{r}{R_s}\right)^{2l+1} \right] \Big/ \left[l + 1 + l \left(\frac{R_0}{R_s}\right)^{2l+1} \right], \end{aligned} \tag{18}$$

where g and h are determined by Equation (15).

2. Issues on Normaliztion

The associated Legendre functions may have different normalization conventions in different cases. From Equation (13), in our scheme we have

$$\int_{-1}^{1} |P_l^m(x)|^2 \mathrm{d}x = \frac{2}{2l+1}(2 - d_0), \qquad d_0 = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}. \tag{19}$$

A more widely used form is

$$\int_{-1}^{1} |\tilde{P}_l^m(x)|^2 \mathrm{d}x = \frac{2(l+m)!}{(2l+1)(l-m)!}, \qquad 0 \le m \le l, \tag{20}$$

with the following properties

$$\tilde{P}_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} \tilde{P}_l^m(x), \tag{21}$$

$$\tilde{P}_l^l(x) = (-1)^l (2l-1)!!(1-x^2)^{l/2}, \tag{22}$$

$$(l-m+1)\tilde{P}_{l+1}^m(x) = (2l+1)x\tilde{P}_l^m(x) - (l+m)\tilde{P}_{l-1}^m(x). \tag{23}$$

Two sets of Legendre functions are related by

$$P_l^m(x) = (-1)^m \sqrt{\frac{(l-m)!}{(l+m)!}} \sqrt{2-d_0}\, \tilde{P}_l^m(x). \tag{24}$$

So in our convention Equation (21)-(23) become

$$P_l^{-m}(x) = (-1)^m P_l^m(x), \tag{25}$$

$$P_l^l(x) = \sqrt{\frac{(2l-1)!!}{(2l)!!}} \sqrt{2-d_0}(1-x^2)^{l/2}, \tag{26}$$

$$\sqrt{(l+1)^2-m^2} P_{l+1}^m(x) = (2l+1)xP_l^m(x) - \sqrt{l^2-m^2} P_{l-1}^m(x). \tag{27}$$

Equation (24) can be used to convert standard Legendre functions for our use. Equation (25)-(27) can be used recursively to generate our own set of Legendre functions.

3. Alternative Method for Computing *g* and *h*

We may alternatively utilize the spherical harmonic expansion result from helioseismology to obtain the g and h coefficients. If the signal on the photosphere at a particular moment is $f(\theta, \phi)$, then

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\phi), \qquad f_l^m \in \mathcal{C}, \tag{28}$$

with the following normalization

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta Y_l^m(\theta,\phi) Y_{l'}^{m'*}(\theta,\phi) = 4\pi \delta_{ll'} \delta_{mm'}. \tag{29}$$

In the most common version,

$$Y_l^{-m} = (-1)^m Y_l^{m*}. \tag{30}$$

Consider Equation (6), (20), (29) and (30), we have

$$Y_l^m(\theta,\phi) = \sqrt{(2l+1)\frac{(l-m)!}{(l+m)!}}\tilde{P}_l^m(\theta)e^{im\phi}, \tag{31}$$

where $\tilde{P}_l^m$ is defined in Equation (20). So we have the following expansion:

$$\begin{aligned} f_l^m &= \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta f(\theta,\phi)Y_l^{m*}(\theta,\phi) \\ &= \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta f(\theta,\phi)\sqrt{(2l+1)\frac{(l-m)!}{(l+m)!}}\tilde{P}_l^m(\theta)e^{-im\phi} \\ &= \frac{(-1)^m}{4\pi}\sqrt{\frac{2l+1}{2-d_0}}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta f(\theta,\phi)P_l^m(\theta)e^{-im\phi}. \end{aligned} \tag{32}$$

Compare this with Equation (15), we find the connection

$$g_{lm} = (-1)^m\sqrt{(2l+1)(2-d_0)}\,\Re(f_l^m), \tag{33}$$

$$h_{lm} = -(-1)^m\sqrt{(2l+1)(2-d_0)}\,\Im(f_l^m). \tag{34}$$

4. Two Other Versions: Global Helioseismology and Solarsoft

As different codes may use different normalizations and algorithms, we need to be careful when using coefficient sets. Here are two other codes the community is using.

Schou's (Schou and Brown, 1994) global helioseismology code is adapted to compute the harmonic expansion coefficient. The synoptic map is considered as a single point time series. FFT is first applied on each row of points and a mask dot product is used to get the coefficient. In this version, f_l^m is given for non-negative m's. The following quations link the result to our g's and h's.

$$g_{lm} = \frac{\sqrt{(2l+1)(2-d_0)}}{2}\,\Re(f_l^m), \tag{35}$$

$$h_{lm} = \frac{\sqrt{(2l+1)(2-d_0)}}{2}\,\Im(f_l^m). \tag{36}$$

The PFSS package in Solar Soft takes another road. The map is first resampled to the optimized Gauss-Legendre grid before computes $f_{I_{lm}}$ and $f_{O_{lm}}$ in Equation (4) and (5). The result satisfies

$$g_{lm} \propto \Re(-l f_{I_{lm}} + (l+1) f_{O_{lm}}), \tag{37}$$

$$h_{lm} \propto \Im(-l f_{I_{lm}} + (l+1) f_{O_{lm}}). \tag{38}$$

The result might be sensitive to the resampling. The constant factor here is yet to be established.

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