Notes on PFSS Extrapolation

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Abstract This is a documentation on the Stanford PFSS model. Detailed mathematical deductions are provided for the use of the model. Some brief documentation on PFSS-like models (SCS, HCCSSS, etc.) is also provided.

1. Basic Equation

The most common version of PFSS model (Hoeksema, 1984; Wang and Sheeley Jr., 1992) currently in use takes global radial Carrington synoptic maps as input. In these maps, photospheric fields are sampled on a heliographic coordinate, evenly spaced either in latitude or sine-latitude steps. If the field is purely potential, we have

$$\vec{B} = -\nabla\Psi, \tag{1}$$

where

$$\nabla^2\Psi = 0. \tag{2}$$

We assume the existence of a spherical "source surface" at a radius of R_s (usually at $2.5R_\odot$), beyond which all field lines are open and radius. The potential arises from both inside the inner boundary R_0 (photsphere, or $R_\odot$) and outside the outer boundary, or the source surface:

$$\Psi = \Psi_I + \Psi_O, \tag{3}$$

with

$$\Psi_I = \sum_{l=0}^{\infty} r^{-(l+1)} \sum_{m=-l}^{l} f_{I_{lm}} Y_{lm}(\theta,\phi), \tag{4}$$

$$\Psi_O = \sum_{l=0}^{\infty} r^{l} \sum_{m=-l}^{l} f_{O_{lm}} Y_{lm}(\theta,\phi). \tag{5}$$

Scale r in terms $R_\odot$ for Ψ_I and in terms of R_s for Ψ_O. Use the fact that

$$Y_{lm}(\theta,\phi) = k_{lm} P_l^m(\cos\theta) e^{im\phi}. \tag{6}$$

The real part of Ψ can be generalized from Equation (3) through (6):

$$\Psi \;=\; R_0 \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\cos\theta) \left\{ g'_{lm}\cos m\phi \left[\left(\frac{R_0}{r}\right)^{l+1} + \frac{R_s}{R_0}\left(\frac{r}{R_s}\right)^{l} c_{lm} \right] \right.$$

$$+ h'_{lm} \sin m\phi \left[\left(\frac{R_0}{r} \right)^{l+1} + \frac{R_s}{R_0} \left(\frac{r}{R_s} \right)^{l} d_{lm} \right] \Bigg\}, \tag{7}$$

where g'_{lm}, h'_{lm}, c_{lm} and d_{lm} are the unknown coefficients. Note that the normalization of the spherical harmonics and associated Legendre fuctions can be tricky. We will simply present the normalization we adopted here and leave the detailed description to Section 2.

By definition, the field lines turn radial at the source surface. This means the field vector is purely radial at R_s, or rather, the potential is a constant on the source surface. Set this potential to 0, we then have

$$c_{lm} = d_{lm} = - \left(\frac{R_0}{R_s} \right)^{l+2} = c_l. \tag{8}$$

Now our sole task is to determine g'_{lm} and h'_{lm}, using the inner boundary condition (photospheric field). Write B_r from Equation (1) specifically:

$$\begin{aligned} B_r(r,\theta,\phi) &= -\frac{\partial \Psi}{\partial r} = \sum_{lm} P_l^m(\cos\theta)(g'_{lm}\cos m\phi + h'_{lm}\sin m\phi) \\ & \qquad \left[(l+1) \left(\frac{R_0}{r} \right)^{l+2} - l \left(\frac{r}{R_s} \right)^{l-1} c_l \right]. \end{aligned} \tag{9}$$

At inner boundary, we have

$$B_r(R_0,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\cos\theta)(g_{lm}\cos m\phi + h_{lm}\sin m\phi), \tag{10}$$

where

$$g_{lm} = g'_{lm} \left[l + 1 + l \left(\frac{R_0}{R_s} \right)^{2l+1} \right], \tag{11}$$

$$h_{lm} = h'_{lm} \left[l + 1 + l \left(\frac{R_0}{R_s} \right)^{2l+1} \right]. \tag{12}$$

Now we make use of the orthogonal property of the associated Legendre function (in our convention)

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta \, P_l^m(\cos\theta) \mathop{}_{\sin}^{\cos} m\phi \, P_{l'}^{m'}(\cos\theta) \mathop{}_{\sin}^{\cos} m'\phi = \frac{4\pi}{2l+1} \delta_{ll'} \delta_{mm'}. \tag{13}$$

Note when $m = 0$ Equation (13) holds for the $\cos m\phi$ case, while the $\sin m\phi$ case simply yields 0. An integration of Equation (10) then shows us how to obtain g

and h. Here, h_{l0} is obviously 0.

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta \, B_r(R_0,\theta,\phi) P_l^m(\cos\theta) \frac{\cos}{\sin} m\phi = \frac{4\pi}{2l+1} \frac{g_{lm}}{h_{lm}}. \tag{14}$$

For a synoptic map $(X \times Y)$ in sine-latitude format, the Equation (14) becomes

$$\begin{pmatrix} g_{lm} \\ h_{lm} \end{pmatrix} = \frac{2l+1}{XY} \sum_{i=1}^{X} \sum_{j=1}^{Y} B_r(R_0,\theta_i,\phi_j) P_l^m(\cos\theta_i) \frac{\cos}{\sin} m\phi_j. \tag{15}$$

Thus the potential is solved. There are other ways to compute g and h, as we will see in Section 3.

To conclude, we have the solutions in the following form.

$$B_r(r,\theta,\phi) = -\frac{\partial\Psi}{\partial r} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\cos\theta)(g_{lm}\cos m\phi + h_{lm}\sin m\phi) \times \left(\frac{R_0}{r}\right)^{l+2} \left[l+1+l\left(\frac{r}{R_s}\right)^{2l+1}\right] \Bigg/ \left[l+1+l\left(\frac{R_0}{R_s}\right)^{2l+1}\right], \tag{16}$$

$$B_\theta(r,\theta,\phi) = -\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{\partial P_l^m(\cos\theta)}{\partial\theta}(g_{lm}\cos m\phi + h_{lm}\sin m\phi) \times \left(\frac{R_0}{r}\right)^{l+2} \left[1-\left(\frac{r}{R_s}\right)^{2l+1}\right] \Bigg/ \left[l+1+l\left(\frac{R_0}{R_s}\right)^{2l+1}\right], \tag{17}$$

$$B_\phi(r,\theta,\phi) = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{mP_l^m(\cos\theta)}{\sin\theta}(g_{lm}\sin m\phi - h_{lm}\cos m\phi) \times \left(\frac{R_0}{r}\right)^{l+2} \left[1-\left(\frac{r}{R_s}\right)^{2l+1}\right] \Bigg/ \left[l+1+l\left(\frac{R_0}{R_s}\right)^{2l+1}\right], \tag{18}$$

where g and h are determined by Equation (15).

At the poles $(\theta = 0, \pi)$, only some (l, m) terms contribute to the solution. This is mainly because $P_l^m(\cos\theta) \propto \sin^m\theta$. For B_r, only the $m = 0$ terms contribute. For B_θ and B_ϕ, only the $m = 1$ terms contribute.

2. Issues on Normalization

The associated Legendre functions may have different normalization conventions in different cases. From Equation (13), in our scheme we have

$$\int_{-1}^{1} |P_l^m(x)|^2 \mathrm{d}x = \frac{2}{2l+1}(2-d_0), \qquad d_0 = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}. \tag{19}$$

A more widely used form is

$$\int_{-1}^{1} |\tilde{P}_l^m(x)|^2 \mathrm{d}x = \frac{2(l+m)!}{(2l+1)(l-m)!}, \qquad 0 \le m \le l, \tag{20}$$

with the following properties

$$\tilde{P}_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} \tilde{P}_l^m(x), \tag{21}$$

$$\tilde{P}_l^l(x) = (-1)^l (2l-1)!!(1-x^2)^{l/2}, \tag{22}$$

$$(l-m+1)\tilde{P}_{l+1}^m(x) = (2l+1)x\tilde{P}_l^m(x) - (l+m)\tilde{P}_{l-1}^m(x). \tag{23}$$

Two sets of Legendre functions are related by

$$P_l^m(x) = (-1)^m \sqrt{\frac{(l-m)!}{(l+m)!}} \sqrt{2-d_0}\,\tilde{P}_l^m(x). \tag{24}$$

So in our convention Equation (21)-(23) become

$$P_l^{-m}(x) = (-1)^m P_l^m(x), \tag{25}$$

$$P_l^l(x) = \sqrt{\frac{(2l-1)!!}{(2l)!!}} \sqrt{2-d_0}(1-x^2)^{l/2}, \tag{26}$$

$$\sqrt{(l+1)^2-m^2}\,P_{l+1}^m(x) = (2l+1)xP_l^m(x) - \sqrt{l^2-m^2}\,P_{l-1}^m(x). \tag{27}$$

Equation (24) can be used to convert standard Legendre functions for our use.

Equations (25)-(27) can be used recursively to generate our own set of Legendre functions. In our implementation (Zhao and Hoeksema, 1994), we start by calculating P_m^m using Equation (26). Then we calculate all terms for fixed m using Equation (27), utilizing the fact that $P_{m-1}^m(x) = 0$.

We also need to calculate $\partial P_l^m(\cos\theta)/\partial\theta$. In our implementation, we have

$$\begin{aligned} \sqrt{(l+1)^2-m^2}\frac{dP_{l+1}^m(x)}{dx} = (2l+1)\left[x\frac{dP_l^m(x)}{dx} - \sqrt{1-x^2}P_l^m(x)\right] \\ -\sqrt{l^2-m^2}\frac{dP_{l-1}^m(x)}{dx}. \end{aligned} \tag{28}$$

3. Alternative Method for Computing g and h

We may alternatively utilize the spherical harmonic expansion result from helioseismology to obtain the g and h coefficients. If the signal on the photosphere at a particular moment is $f(\theta,\phi)$, then

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\phi), \qquad f_l^m \in \mathcal{C}, \tag{29}$$

with the following normalization

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta Y_l^m(\theta,\phi) Y_{l'}^{m'*}(\theta,\phi) = 4\pi\delta_{ll'}\delta_{mm'}. \tag{30}$$

In the most common version,

$$Y_l^{-m} = (-1)^m Y_l^{m*}. \tag{31}$$

Consider Equation (6), (20), (30) and (31), we have

$$Y_l^m(\theta,\phi) = \sqrt{(2l+1)\frac{(l-m)!}{(l+m)!}}\tilde{P}_l^m(\cos\theta)e^{im\phi}, \tag{32}$$

where $\tilde{P}_l^m$ is defined in Equation (20). So we have the following expansion:

$$\begin{aligned} f_l^m &= \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta f(\theta,\phi) Y_l^{m*}(\theta,\phi) \\ &= \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta f(\theta,\phi) \sqrt{(2l+1)\frac{(l-m)!}{(l+m)!}}\tilde{P}_l^m(\cos\theta)e^{-im\phi} \\ &= \frac{(-1)^m}{4\pi}\sqrt{\frac{2l+1}{2-d_0}}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta f(\theta,\phi) P_l^m(\cos\theta)e^{-im\phi}. \end{aligned} \tag{33}$$

Compare this with Equation (15), we find the connection

$$g_{lm} = (-1)^m\sqrt{(2l+1)(2-d_0)}\,\Re(f_l^m), \tag{34}$$

$$h_{lm} = -(-1)^m\sqrt{(2l+1)(2-d_0)}\,\Im(f_l^m). \tag{35}$$

4. Two Other Versions: Global Helioseismology and Solarsoft

As different codes may use different normalizations and algorithms, we need to be careful when using coefficient sets. Here are two other codes the community is using.

Schou's (Schou and Brown, 1994) global helioseismology code is adapted to compute the harmonic expansion coefficient. The synoptic map is considered as a single point time series. FFT is first applied on each row of points and a mask dot product is used to get the coefficient. In this version, f_l^m is given for non-negative m's. The following quations link the result to our g's and h's.

$$g_{lm} = \frac{\sqrt{(2l+1)(2-d_0)}}{2}\,\Re(f_l^m), \tag{36}$$

$$h_{lm} = \frac{\sqrt{(2l+1)(2-d_0)}}{2}\,\Im(f_l^m). \tag{37}$$

The PFSS package in Solar Soft takes another road. The map is first resampled to the optimized Gauss-Legendre grid before computes $f_{I_{lm}}$ and $f_{O_{lm}}$ in Equation (4) and (5). The result satisfies

$$g_{lm} \propto \Re(-l f_{I_{lm}} + (l+1) f_{O_{lm}}), \tag{38}$$

$$h_{lm} \propto \Im(-l f_{I_{lm}} + (l+1) f_{O_{lm}}). \tag{39}$$

The result might be sensitive to the resampling. The constant factor here is yet to be established.

5. PFSS-Like Models

The effect of currents are included in several other PFSS-based models. Schatten (Schatten, 1971) proposed a model called potential-field-current-sheet (PFCS) that includes the effect of the current sheets in streamers. Zhao and Hoeksema (Zhao and Hoeksema, 1994) provided a modified horizontal-current-current-sheet model (HCCS) that introduces the effect of horizontal current, based on Bogdan and Low's (Bogdan and Low, 1986) magnetic-static solution. Later, Zhao and Hoeksema (Zhao and Hoeksema, 1995) revised the HCCS model, adding a source surface at a higher altitude. This new model is called horizontal-current-current-sheet-source-surface (HCCSSS) model.

In brief, these PFSS-like models divide the solar atmosphere into different regions. Below a spherical surface (cusp surface) the field is more potential-like, without the effect of current sheet. Above this surface the current sheet comes in. HCCSSS model an extra source surface at higher up opens all field to be radial. Both the HCCS and HCCSSS models have a horizontal current flowing everywhere.

We generalize the algorithm as following (Zhao and Hoeksema, 1994; Zhao and Hoeksema, 1995). First, the field every where can be computed from a potential function Ψ:

$$\Psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} R_l(r) P_l^m(\cos\theta)(g_{lm}\cos m\phi + h_{lm}\sin m\phi), \tag{40}$$

and fields are computed by

$$\vec{B} = -\eta(r)\frac{\partial\Psi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}\hat{\phi}, \tag{41}$$

Table 1. Various form of $R_l(r)$ in Different Models for Equation (40)

Model	Lower Region	Upper Region
PFCS	$\frac{R_0(\frac{R_0}{r})^{l+1}[1-(\frac{r}{R_s})^{2l+1}]}{l+1+l(\frac{R_0}{R_s})^{2l+1}}$	$R_c^2\frac{(R_c+a)^l}{(l+1)(r+a)^{l+1}}$
HCCS	$R_0^2\frac{(R_0+a)^l}{(l+1)(r+a)^{l+1}}$	$R_c^2\frac{(R_c+a)^l}{(l+1)(r+a)^{l+1}}$
HCCSSS	$R_0^2\frac{(R_0+a)^l}{(l+1)(r+a)^{l+1}}$	$R_c^2\frac{\frac{1}{(r+a)^{l+1}}-\frac{(r+a)^l}{(R_s+a)^{2l+1}}}{\frac{l+1}{(R_c+a)^l}+\frac{l(R_c+a)^{l+1}}{(Rs+a)^{2l+1}}}$
PFSS	$R_0\frac{(\frac{R_0}{r})^{l+1}[1-(\frac{r}{R_s})^{2l+1}]}{l+1+l(\frac{R_0}{R_s})^{2l+1}}$	

where

$$\eta(r) = (1+\frac{a}{r})^2, \tag{42}$$

in which a parameterizes the length scale of horizontal eletric current in the corona. Note when $a=0$ we simply return to models without horizontal currents (PFSS).

The potential is determined by boundary conditions on the imaginary spherical surfaces that we use to divide solar atmosphere. (1) To get the spherical harmonics (g_{lm} and h_{lm} in Equation (40)), photospheric field is decomposed for the lower region (below cusp surface). Field value is then computed at the upper boundary of the lower region (cusp surface), where Schatten's least-square technique (Schatten, 1971) is used to get the coefficient for the higher region. (2) The radial part (Φ in Equation (40)) is determined by the assumption we make of the field on the boudaries: whether it is radial, etc. To summerize, we put the form of Φ function in Table 1.

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